Schur duality

Laura Mancinska

August 1, 2008

1 Introduction

In this report we will first, present the result from representation theory known as *Schur duality* and second, discuss its applications to quantum information theory. The original motivation (of the author of this review) for studying this result was to understand the unitaries that commute with qudit permutations (see Section 4.1). However, in the quantum information research community the main reason for interest in Schur duality is the fact that it can be used to accomplish many information theoretic tasks (see Section 4.2). The main reference for this survey is Chapter 5 of Aram Harrow's thesis [1].

2 Representation theory background

A *representation* of a group G is a pair $(\phi, \mathbb{C}^n)$, where $\phi : G \to \mathrm{GL}(n, \mathbb{C})$ is a homomorphism and $\mathrm{GL}(n, \mathbb{C})$ is the set of invertible $n \times n$ complex matrices. We refer to $\mathbb{C}^n$ as the *representation space* of ϕ and to n as its *dimension*, denoted by $\dim \phi$.

Let $(\phi_1, \mathbb{C}^{n_1})$ and $(\phi_2, \mathbb{C}^{n_2})$ be representations of a group G. Then we can obtain another two representations of G by taking their *direct sum* $(\phi_1 \oplus \phi_2, \mathbb{C}^{n_1+n_2})$ and *tensor product* $(\phi_1 \otimes \phi_2, \mathbb{C}^{n_1 n_2})$, given by

$$\phi_1 \oplus \phi_2(g) = \phi_1(g) \oplus \phi_2(g) \qquad \phi_1 \otimes \phi_2(g) = \phi_1(g) \otimes \phi_2(g) \qquad \forall g \in G$$

We say that $(\phi, \mathbb{C}^n)$ is *irreducible* if it cannot be decomposed into a direct sum of at least two other representations. With this definition we see that all 1-dimensional representations are irreducible. It can also be shown that any representation of a finite group G in some basis can be expressed as a direct sum of irreducible representations of G. Another important result known as *Schur's lemma* says that if $(\phi_1, \mathbb{C}^{n_1})$ and $(\phi_2, \mathbb{C}^{n_2})$ are two irreducible representations of G and M is a $\dim(\phi_1) \times \dim(\phi_2)$ matrix satisfying $\phi_1(g)M = M\phi_2(g)$ for all $g \in G$, then either $M = 0$ or $M = cI$, for some $c \in \mathbb{C}$. For more details on basic representation theory consult [3].

3 Schur duality

Consider the following two representations

1. $\left(\mathbf{Q}, \left(\mathbb{C}^d\right)^{\otimes n}\right)$ of $\mathcal{U}(d)$, where $\mathbf{Q}(U)\left|i_1 i_2 \ldots i_n\right\rangle = U\left|i_1\right\rangle U\left|i_2\right\rangle \ldots U\left|i_n\right\rangle \; \forall U \in \mathcal{U}(d)$

2. $\left(\mathbf{P}, \left(\mathbb{C}^d\right)^{\otimes n}\right)$ of S_n, where $\mathbf{P}(\pi)\left|i_1 i_2 \ldots i_n\right\rangle = \left|i_{\pi^{-1}(1)}\right\rangle \left|i_{\pi^{-1}(2)}\right\rangle \ldots \left|i_{\pi^{-1}(n)}\right\rangle \; \forall \pi \in S_n$

Since $\mathbf{Q}$ and $\mathbf{P}$ commute, we can define representation $\left(\mathbf{QP}, \left(\mathbb{C}^d\right)^{\otimes n}\right)$ of $U(d) \times S_n$ as

$$\mathbf{QP}(U, \pi) := \mathbf{Q}(U)\mathbf{P}(\pi) = \mathbf{P}(\pi)\mathbf{Q}(U) \qquad \forall (U, \pi) \in U(d) \times S_n$$

Theorem 1. (*Schur duality*) There exist a basis, known as *Schur basis*, in which representation $\left(\mathbf{QP}, \left(\mathbb{C}^d\right)^{\otimes n}\right)$ of $\mathcal{U}(d) \times S_n$ decomposes into irreducible representations[1] $\mathbf{q}_\lambda$ and $\mathbf{p}_\lambda$ of $\mathcal{U}(d)$ and S_n respectively:

$$\mathbf{QP}(U, \pi) \cong \bigoplus_{\lambda \in \mathrm{Par}(n,d)} \mathbf{q}_\lambda(U) \otimes \mathbf{p}_\lambda(\pi) \tag{1}$$

We use $\mathrm{Par}(n, d)$ to denote the set of partitions of n into d parts. Note that partitions are order independent, i.e., $(2, 0)$ and $(0, 2)$ are in fact the same partition.

In order to prove the above theorem, we first observe that algebras generated by $\mathbf{P}$ and $\mathbf{Q}$ centralize each other. Then we can apply *double commutant theorem* to get expression (1) only with unspecified range of λ. In order to specify the range, we find a correspondence between irreducible representations of S_n and U_d and partitions $\mathrm{Par}(n, d)$. This was only a very rough outline, for the complete proof (and also double commutant theorem) see [4].

We call the unitary transformation performing the basis change from standard basis to Schur basis, *Schur transform* and denote by U_{sch}. It has been shown in [2] that Schur transform can be implemented efficiently on a quantum computer. This gives us efficient algorithms for tasks considered in Section 4.2.

4 Applications

4.1 Unitaries commuting with qudit permutations

Note that $P_\pi = \mathbf{QP}(I, \pi)$, where $P_\pi \in \mathcal{U}(d^n)$ permutes qudits according to π. Now we apply Schur duality to get

$$P_\pi = \mathbf{QP}(I, \pi) \cong \bigoplus_{\lambda \in \mathrm{Par}(n,d)} \mathbf{q}_\lambda(I) \otimes \mathbf{p}_\lambda(\pi) = \bigoplus_{\lambda \in \mathrm{Par}(n,d)} I_{\dim(\mathbf{q}_\lambda)} \otimes \mathbf{p}_\lambda(\pi) \tag{2}$$

In the last equality we use the fact that evaluating any representation at group identity gives identity matrix. Equation (2) shows that permutation matrices P_π are block diagonalised in Schur basis. Consider family of unitaries given by

$$\mathcal{F} := \bigoplus_{\lambda \in \mathrm{Par}(n,d)} \mathcal{U}\left(\dim(\mathbf{q}_\lambda)\right) \otimes I_{\dim(\mathbf{p}_\lambda)} \tag{3}$$

[1] More precisely, for $\mathcal{U}(d)$ there is an additional constraint that the irreducible representations must be *polynomial*, i.e., the matrix entries of $\mathbf{q}_\lambda(U)$ are polynomial functions of U_{ij}.

in Schur basis. Due to the decomposition of qudit permutation matrices in Schur basis given in equation (2), it is obvious that unitaries of the form (3) commute with all qudit permutations. On the other hand, since representations $\mathbf{p}_\lambda$ in equation (2) are irreducible via Schur's lemma we get that any unitary that commutes with all qudit permutation has to have identity matrix in the blocks corresponding to $\mathbf{p}_\lambda$. This shows that $\mathcal{F}$ is *exactly* the set of unitaries commuting with qudit permutations. If we want a description of $\mathcal{F}$ in standard basis we have to conjugate each matrix with Schur transform.

4.2 Applications to quantum information theory

In classical information theory method of types can be used to perform such task as estimating probability distribution, randomness concentration and data compression (see [5]). It has been shown that Schur basis can be used to generalize classical method of types, thus enabling us to perform quantum analogues of the previously mentioned tasks. In fact Schur basis is a natural choice if we want to study systems with permutation symmetry. For example, in case of n copies of mixed state $\rho \in L(d)$ we have:

$$U_{\text{sch}} \rho^{\otimes n} U_{\text{sch}}^{\dagger} = \bigoplus_{\lambda \in \text{Par}(n,d)} \mathbf{q}_\lambda(\rho) \otimes I_{\dim(\mathbf{p}_\lambda)} \tag{4}$$

Schur transform can be used to perform the following tasks:

- *Estimation of the spectrum of an unknown mixed state $\rho \in L(\mathbb{C}^d)$ from $\rho^{\otimes n}$*
  If we apply Schur transform to $\rho^{\otimes n}$ and measure the label $\lambda \in \text{Par}(n, d)$, then a good estimate of the spectrum of ρ is given by $(\lambda_1/n, \ldots, \lambda_d/n)$.

- *Universal distortion-free entanglement concentration using only local operations*
  Alice and Bob share n copies of some unknown partially entangled state $|\psi\rangle_{AB} \in \mathbb{C}^{2d}$ and the task is to produce maximally entangled states using only local operations. If Alice and Bob apply Schur transform to their n halves of $|\psi\rangle$, measure the label $\lambda \in \text{Par}(n, d)$, discard representation the space of $\mathbf{q}_\lambda$, then they are left with perfectly entangled state in the representation space of $\mathbf{p}_\lambda$.

- *Encoding into decoherence free subsystems*
  If we know that noise will act identically on each of the n systems, information can be protected from decoherence by encoding it into subspace corresponding to irreducible representations of S_n, since the noise acts trivially in this subspace.

Another applications of Schur transform include communication without shared reference frame and universal compression of quantum data.

References

[1] A. Harrow. Applications of coherent classical communication and the Schur transform to quantum information theory, 2001

[2] D. Bacon, I.Chuang, A.Harrow. The quantum Schur Transform: Efficient Qudit Circuits, Phys. Rev. A, 2002

[3] J. P. Serre. Linear Representations of Finite Groups, Springer, 1977.

[4] R. Goodman and N.R. Wallach. Representations and Invariants of the Classical Groups, Cambridge University Press, 1998

[5] T. M. Cover,J. A. Thomas. Elements of Information Theory. Series in Telecommunication. John Wiley and Sons, 1991